

63 4-2

CATALOGED BY DDC  
AS AD No. 408749

(SP Series)



---

SP-1216/000/00

Optimal Strategies for Maximum-Number Games

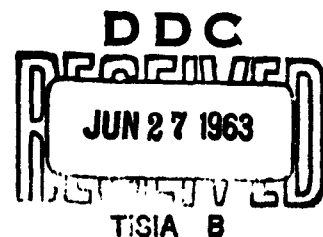
by

Milton Ash and Wayne Jones

22 May 1963

---

SYSTEM DEVELOPMENT CORPORATION, SANTA MONICA, CALIFORNIA



22 May 1963

1

SP-1216/000/00

## Optimal Strategies for Maximum-Number Games

by

Milton Ash and Wayne Jones

Recently the following problem,<sup>(1)</sup> which is but one of a genre of like problems<sup>(2)(3)</sup>, was posed: Given  $N$  different real numbers (e.g.,  $N=1000$ ) on cards face down on a table, what is the stopping rule so that the probability of choosing the largest number is maximized? The additional rules of the game are: (a) that as the numbers are chosen, the cards are turned up so that a comparison of the discards can be made with the immediate choice, and (b) that only the immediate choice can be declared to be largest.

First define two state subscripts: State  $k >$  : a choice has just been made, with  $k$  of  $N$  choices remaining, and this immediate choice happens to be the largest of those chosen thus far. State  $k <$  : ditto, except that this immediate choice doesn't happen to be the largest of those chosen thus far.

Let  $P_{k>}$  be the probability of being in state  $k >$  and, if one stops, being declared the winner, i.e., holding the largest of the  $N$  numbers.  $P_{k<}$  is defined in the same way for the state  $k <$ . Then the following stopping rule, "GO until  $N-k_0$  are chosen; then stop if in state  $k_0 >$ ," which maximizes  $P_{k_0 >}$  can be obtained from the following two equations of state. These are written by applying the principle of optimality of dynamic programming, and are given by:

$$\begin{aligned}
 (1) \quad P_{k>} &= \max_k \begin{cases} \text{If STOP: } \frac{N-k}{N} \\ \text{If GO : } \left(\frac{1}{N+1-k}\right)P_{k-1>} + \left(1 - \frac{1}{N+1-k}\right)P_{k-1<} \end{cases} ; P_{0>} = 1 \\
 (2) \quad P_{k<} &= \max_k \begin{cases} \text{If STOP: } 0 \\ \text{If GO : } \left(\frac{1}{N+1-k}\right)P_{k-1>} + \left(1 - \frac{1}{N+1-k}\right)P_{k-1<} \end{cases} ; P_{0<} = 0
 \end{aligned}$$

If one begins to choose numbers,  $k > k_0$ , the "If GO" > "If STOP" quantities on the right side of equation (1) will hold, until for some  $k=k_0$  "If GO" = "If STOP." For  $k < k_0$ , "If GO < If STOP." The stopping rule is to GO until  $N-k_0$  numbers are chosen. If in state  $k_0 >$ , then STOP. If not, keep choosing until the first  $k >$  state is reached for  $k < k_0$ , then STOP. Again, note that  $k$  designates the choices remaining.

To find  $k_0$ , equation (1) at  $k=k_0$ , which is the turning point in the choice sequence from GO to STOP as  $k$  decreases, yields

$$(3) \quad 1 - \frac{k_0}{N} = \frac{1}{N+1-k_0}P_{k_0-1>} + \left(1 - \frac{1}{N+1-k_0}\right)P_{k_0-1<}$$

Eliminating  $P_{k_0-1<}$  by iteration using equation (2) gives

$$(4) \quad 1 - \frac{k_0}{N} = \frac{P_{k_0-1>}}{N+1-k_0} + (N-k_0) \sum_{j=1}^{k_0-1} \frac{P_{j-1>}}{(N-j)(N+1-j)}$$

Using the fact that  $P_{k_0-j>} = \frac{N+j-k_0}{N}$ ,  $k_0$  is the (smaller) root of

22 May 1963

3

SP-1216/000/00

$$(5) \quad (1 - \frac{k_0}{N})(1 - \sum_{j=1}^{k_0-1} \frac{1}{N-j}) = \frac{1}{N}$$

If  $N$  is large (like 1000),  $\frac{1}{N}$  above is replaced by zero, and  $k_0$  is given to good approximation by

$$(6) \quad \int_1^{k_0} \frac{dx}{N-x} \approx 1$$

Then  $k_0 \approx (1 - e^{-1})N$  and  $N - k_0 = N/e \approx 0.368N$ . The stopping rule for  $N=1000$  is to discard randomly, stopping at the 368th number if it is the largest of the discards thus far. If not, choose the first number after the 368th that is the largest and then stop.  $P_{k_0 >} = e^{-1}$ . From equations (1) and (2), for  $k > k_0$  and (GO),  $P_{k >} = P_{k <}$ . This implies  $P_{k_0 >} = P_{k+1 >} = \dots = P_{N >}$ . Then the probability of winning with the above stopping rule is  $e^{-1}$ .

For an added fillip, assume that it costs  $C$  dollars per choice, and that picking the largest number  $x$  yields a prize in the amount of  $x$  dollars plus a bonus of  $\frac{B}{x}$  dollars (or an additional cost of  $\frac{B}{x}$  dollars corresponding to a house cut). Then what is the stopping rule to maximize the expectation of winning?

The upper right side of (1) now reads: If STOP:  $(\frac{N-k}{N})(x \pm \frac{B}{x} - NC)$ . The rest remains the same, giving, for large  $N$ ,  $k_0 = [1 - \exp(-(x \pm \frac{B}{x} - NC))]N$ , which reduces to the old game for  $x=1$ ,  $B=C=0$ . This has obvious interesting ramifications for various sizes of the largest number  $x$ , ante  $C$ , bonus

22 May 1963

<sup>4</sup>  
(Last Page)

SP-1216/000/00

constant  $\pm B$ , and the total number of cards  $N$ . For example, if  $x$  is very large or small compared to  $NC$ , and if  $B > 0$  and such that  $x + \frac{B}{x} - NC$  is always positive, then  $k_0 \rightarrow N$ , so that it pays to stop early.

- (1) B. H. Bissinger, C. Siegel, American Mathematical Monthly, Vol. 70, No. 3, p. 336, March 1963.
- (2) M. Sakaguchi "Dynamic Programming of Some Sequential Sampling Design," Jo. Math. Analysis and Applications, Vol. 2, No. 3, pp. 446-466, June 1961.
- (3) B. Gluss "A Note on a Computational Approximation to the Two Machine Problem," Jo. Information and Control, Vol. 1, pp. 268-275, 1957.

UNCLASSIFIED

System Development Corporation,  
Santa Monica, California  
OPTIMAL STRATEGIES FOR MAXIMUM-  
NUMBER GAMES.  
Scientific rept., SP-1216/000/00,  
by M. Ash, W. Jones. 22 May 1963,  
4p., 3 refs.

Unclassified report

DESCRIPTORS: Mathematical Analysis.

UNCLASSIFIED

---